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Fuel-Optimal Rendezvous for Linearized Equations of Motion

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A new transformation of the state variables is presented so that some recent necessary and sufficient conditions for solution of the general fuel-optimal linear problem can be efficiently applied. Additional insight into the solution of the boundary-value problem may be obtained by working in this transformed state space. The necessity of inverting a fundamental matrix solution is avoided in this approach. This theory is applied to the problem of rendezvous of a spacecraft near an arbitrary Keplerian orbit in fixed time. Bounded-thrust fuel-optimal spacecraft trajectories are compared for elliptical, parabolic, and hyperbolic orbits.

I. Introduction

IN some cases, fuel-optimal trajectory and rendezvous problems can be simplified through the use of linearized equations about a nominal orbit. Some of the studies in which the nominal orbit is a circle were performed by Tschauner and Hempel,[1] Edelbaum,[2] Prussing,[3,4] Jones,[5] Jezewski and Donaldson,[6] Jezewski,[7] Carter,[8] and, more recently, by Prussing and Clifton,[9] and Larson and Prussing.[10] Studies relative to elliptical nominal orbits were conducted by De Vries,[11] Tschauner and Hempel,[12] Shulman and Scott,[13] Tschauner,[14] Euler and Shulman,[15] Euler,[16] Weiss,[17] and Wolfsberger et al.[18] Both elliptical and circular orbits were considered by Marec,[19] and other references can be found therein. General Keplerian orbits were discussed by Carter and Humi[20] and Carter.[21,22]

Some of the aforementioned work applies to more general linear systems. For example, Prussing and Clifton[9] presented a set of necessary and sufficient conditions for optimal impulsive transfer using general linear equations. Earlier work on the linear problem was done in great generality by Neustadt.[23,24] Recently, Carter[25] has contributed to the problem of fixed-time optimal impulsive trajectories for general linear systems, and Carter and Brient[26] have contributed to the related linear bounded-thrust problem with application to nominal orbits that are Keplerian.

We extend and apply the preceding studies[25,26] here by effecting a transformation from the original state vector to a new pseudostate vector. The optimization problem, recent necessary and sufficient conditions for solution, and the boundary-value problem can be presented in terms of this pseudostate vector without adding to the difficulty of the problem.

The advantage of this transformation is that the new pseudostate vector is constant unless the spacecraft thrustors are activated. The effects of the controls on the transformed state are therefore more apparent to an observer than their effects on the original state. For this reason more insight should be available to the investigator into the optimization problem and its solution. It is not our purpose in this paper to present specific numerical solutions to the resulting boundary-value problem, but we do present the boundary-value problem as a compact set of nonlinear equations that are amenable to various known computational methods.

In the body of this paper, first we present a section that includes some useful results from the theory of linear differential equations. These results show us a way to avoid the problem of inverting a fundamental matrix solution in constructing solutions to optimal trajectory and rendezvous problems. This is especially useful for systems where the coefficients are not constant or where the fundamental matrix solution is not symplectic. In the next section, we present a transformation of the state vector in the context of first impulsive and then bounded-thrust problems. In the final section, the work is applied to the problem of fixed-time optimal maneuvers of a spacecraft near a general Keplerian orbit. A generalized fundamental matrix solution of the adjoint system is presented for this problem. For bounded-thrust, fixed-time, and fixed-end conditions, trajectories and switching functions are displayed and compared for a spectrum of nominal orbits ranging from near circular through hyperbolic.

II. Unforced Trajectories Based on Linear Equations

We shall consider spacecraft trajectory problems in which the equations of motion have been linearized. Whether the problem is modeled as an impulsive thrust problem or a bounded-thrust problem, the equations of motion during a coasting (unpowered) interval define a homogeneous linear differential equation:

$$y'(\theta) = A(\theta)y(\theta) \tag{1}$$

where the prime represents differentiation with respect to the independent variable θ that is contained on a closed bounded interval Θ of real numbers for each $\theta \in \Theta$, $y(\theta) \in \Re^{2m}$ where m is a positive integer, and $A(\theta)$ is a $2m \times 2m$ matrix whose entries are continuous functions of θ. In most applications, the state vector $y(\theta)$ consists of a generalized position and velocity, each in $\Re^m$. For this reason we usually have $m = 3$ or, for problems restricted to an orbital plane, $m = 2$.

It is well known that the complete solution of Eq. (1) is given by

$$y(\theta) = \Phi(\theta)c \tag{2}$$

where $\Phi(\theta)$ is any fundamental matrix solution associated with Eq. (1) and c is an arbitrary constant vector in $\Re^{2m}$. To compute optimal impulsive or bounded-thrust trajectories using some recent results,[25,26] one needs the inverse of this fundamental matrix solution. The primary reason that one needs the inverse of this matrix function is that optimal solutions to the impulsive or bounded-thrust problem can be determined

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through the use of the well-known primer vector discovered by Lawden. This primer vector function q can be described[25,26] by the relationship

$$q(\theta) = R(\theta)^T\lambda \tag{3}$$

where λ is a constant vector in $\Re^{2m}$ and $R(\theta)$ is the $2m \times m$ matrix consisting of the right half of the columns of $\Phi(\theta)^{-1}$. Although Gaussian reduction will allow one to numerically compute the inverse of $\Phi(\theta)$ at a particular θ, one needs an analytical expression for $\Phi(\theta)^{-1}$ that is valid everywhere on the interval Θ. It is known that for systems where $A(\theta)$ is constant, an analytical expression for the inverse of $\Phi(\theta)$ can easily be obtained from $\Phi(\theta)$. First we normalize at 0, that is, we define $\phi(\theta) = \Phi(\theta)\Phi(0)^{-1}$. [We can use Gaussian reduction to invert $\Phi(0)$.] We then use the known relationship $\phi(\theta)^{-1} = \phi(-\theta)$. The result is that $\Phi(\theta)^{-1} = \Phi(0)^{-1}\Phi(-\theta)(0)^{-1}$. It is also known that an expression for $\Phi(\theta)^{-1}$ is easily obtained if the fundamental matrix solution $\Phi(\theta)$ is symplectic.

There are problems where these methods do not apply. The Tschauner-Hempel problem[11,12] is an important example of such a problem. For problems in which the continuous matrix $A(\theta)$ has very general properties, we present the following method of determining the primer vector function (3) without inverting a fundamental matrix solution.

We replace the linear system (1) by its adjoint system:

$$l'(\theta) = -A(\theta)^T l(\theta) \tag{4}$$

where the superscript T denotes the transpose of a matrix or vector. We then find a fundamental matrix solution $\Psi(\theta)$ associated with Eq. (4). It is known from the theory of linear differential equations that $\Psi(\theta)^T = \Phi(\theta)^{-1}$ where $\Phi(\theta)$ is some fundamental matrix solution of Eq. (1). For this reason we can use the m right-hand columns of $\Psi(\theta)^T$ to define the $2m \times m$ matrix $R(\theta)$ and thus determine the primer vector (3). With this method there is no need to find a fundamental matrix solution $\Phi(\theta)$ associated with Eq. (1). The optimal maneuvers are defined by $\Psi(\theta)$ and the boundary conditions.

III. Transformation of the State Vector

We present a transformation of the state vector $y(\theta)$ to a new vector $z(\theta)$ that simplifies the optimization problem formulation and can provide insight into solution of the boundary-value problem. We are given initial conditions

$$y(\theta_0) = y_0 \tag{5}$$

and terminal conditions

$$y(\theta_f) = y_f \tag{6}$$

of the optimal spacecraft trajectory problem where $y_0 = (x_0^T, v_0^T)^T$, $y_f = (x_f^T, v_f^T)^T$, and x_0, x_f, v_0, $v_f \in \Re^m$. The interval $\theta_0 \le \theta \le \theta_f$ defines the closed bounded interval Θ. We let $\Psi(\theta)$ denote any fundamental matrix solution of the adjoint system (4) and define the new pseudostate vector by the transformation

$$z(\theta) = \Psi(\theta)^T y(\theta) - \Psi(\theta_0)^T y_0 \tag{7}$$

If we define the point $z_f \in \Re^{2m}$ by

$$z_f = \Psi(\theta_f)^T y_f - \Psi(\theta_0)^T y_0 \tag{8}$$

then Eqs. (5) and (6) define the following end conditions on z:

$$z(\theta_0) = 0, \qquad z(\theta_f) = z_f \tag{9}$$

The advantage of this transformation is that the new variable $z(\theta)$ changes directly as a result of application of the engine thrustors and is constant during unpowered intervals. To see this, we differentiate Eq. (7), observe from Eq. (4) that $\Psi'(\theta) = -A(\theta)^T\Psi(\theta)$, and note that Eq. (1) applies during coasting intervals. The result is that $z'(\theta) = 0$ identically on coasting intervals.

Of interest also is the scalar valued variable $\zeta(\theta) = -\lambda^T z(\theta)$ where λ is the vector associated with Eq. (3). It will be shown that this variable is related to the fuel consumption of the spacecraft. Multiplying (7) on the left by $-\lambda^T$, we obtain $\zeta(\theta) = -l(\theta)^T y(\theta) + l(\theta_0)^T y_0$ where $l(\theta)$ satisfies the adjoint system (4).

A. Impulsive Problem

We shall consider the state vector to consist of a generalized position $x(\theta) \in \Re^m$ and generalized velocity $v(\theta) \in \Re^m$ so that $y(\theta) = [x(\theta)^T, v(\theta)^T]^T$ for each $\theta \in \Theta$. Given a positive integer n, we consider a finite set $K = \{\theta_1, \ldots, \theta_n\} \subseteq \Theta$ of points of possible discontinuity of $v(\theta)$. Some or all of the points in K are specified (fixed), and the remaining unspecified points are determined through the optimization process. We always specify $\theta_1 = \theta_0$ and $\theta_n = \theta_f$. The notation $|\ |$ is used to indicate the Euclidean norm or magnitude of a vector.

The linear impulsive minimization problem can be stated as follows:

Find the unspecified points in K and the n-velocity increments $\Delta v_i \in \Re^m$ $(i = 1, \ldots, n)$ to minimize the total characteristic velocity

$$\sum_{i=1}^{n} |\Delta v_i|$$

of a spacecraft subject to the differential equation (1) that holds for $\theta \in \Theta$ and $\theta \notin K$, the initial conditions (5), the terminal conditions (6), and the following one-sided limits:

$$\lim_{\theta \to \theta_0 +} v(\theta) = v_0 + \Delta v_1, \qquad \lim_{\theta \to \theta_f -} v(\theta) = v_f - \Delta v_n$$

$$\lim_{\theta \to \theta_i +} v(\theta) = \lim_{\theta \to \theta_i -} v(\theta) + \Delta v_i \qquad (i = 2, \ldots, n-1) \tag{10}$$

We shall show how this problem statement can be formulated in terms of the transformation (7).

Proceeding in a way similar to previous work,[25] we observe that y is a solution of Eq. (1) if and only if for θ, $\theta_i \in \Theta$, we have

$$y(\theta) = \Phi(\theta)\Phi(\theta_i)^{-1} y(\theta_i)$$

Introducing a jump discontinuity of the type in Eq. (10) at θ_i, this becomes

$$y(\theta) = \Phi(\theta)[\Phi(\theta_i)^{-1} y(\theta_i) + R(\theta_i)\Delta v_i], \qquad \theta_i < \theta < \theta_{i+1}$$

Writing $y(\theta_i)$ in terms of all previous jump discontinuities of the form of Eq. (10), we obtain

$$y(\theta) = \Phi(\theta)\Phi(\theta_0)^{-1} y_0 + \Phi(\theta) \sum_{j=1}^{i} R(\theta_j)\Delta v_j, \qquad \theta_i < \theta < \theta_{i+1}$$

This can be written in the form

$$\Psi(\theta)^T y(\theta) - \Psi(\theta_0)^T y_0 = \sum_{j=1}^{i} R(\theta_j)\Delta v_j, \qquad \theta_i < \theta < \theta_{i+1}$$

suggesting the transformation (7). In terms of the new variable $z(\theta)$ on Θ, we have the result

$$z(\theta) = \sum_{j=1}^{i} R(\theta_j)\Delta v_j, \qquad \theta_i \le \theta < \theta_{i+1}$$

$$\text{or} \qquad \theta = \theta_n \qquad \text{if} \qquad i = n \ (i = 1, \ldots, n) \tag{11}$$

The reader should note that the variable z is piecewise constant. For this reason it is advantageous to use the vector $z(\theta)$ rather than the state vector $y(\theta)$. Linear problems can be formulated and solved in terms of z rather than y. More insight into the boundary-value problem should be available because z is piecewise constant.

The linear impulsive minimization problem is now restated as follows:

Find the unspecified points in K and the n-velocity increments $\Delta v_i \in \Re^m$ $(i = 1, \dots, n)$ to minimize the total characteristic velocity

$$\sum_{i=1}^{n} |\Delta v_i|$$

of a spacecraft subject to Eq. (11), and the initial and terminal conditions of Eq. (9).

Necessary and sufficient conditions for solution of the linear impulsive minimization problem where all of the points in K are unspecified were presented by Prussing and Clifton[9] using primer vector theory. Here we follow Ref. 25 where it was found necessary that there exists a vector $\lambda \in \Re^{2m}$ such that

$$\Delta v_i = -q(\theta_i)\alpha_i \qquad (i = 1, \dots, n) \tag{12}$$

$$\alpha_i = 0 \quad \text{or} \quad \lambda^T R(\theta_i) R(\theta_i)^T \lambda = 1 \quad (i = 1, \dots, n) \tag{13}$$

and

$$\alpha_i = 0 \quad \text{or} \quad \lambda^T R'(\theta_i) R(\theta_i)^T \lambda = 0 \tag{14}$$

for the unspecified values of θ_i in K, where in each case $\alpha_i = |\Delta v_i|$. Substituting Eq. (12) into Eq. (11) utilizing Eq. (3), we obtain

$$z(\theta) = -\sum_{j=1}^{i} R(\theta_j) R(\theta_j)^T \lambda \alpha_j, \qquad \theta_i \le \theta < \theta_{i+1} \tag{15}$$

The end conditions (9) establish

$$-\sum_{j=1}^{n} R(\theta_j) R(\theta_j)^T \lambda \alpha_j = z_f \tag{16}$$

The vector λ, the unspecified values of θ_i, and $\alpha_1, \dots, \alpha_n$ are determined through the solution of Eqs. (13), (14), and (16). Additional necessary conditions are obtained from the fact that

$$\alpha_i \ge 0 \qquad (i = 1, \dots, n) \tag{17}$$

and multiplying Eq. (16) by λ^T using Eqs. (13) and (8), we obtain

$$\sum_{j=1}^{n} \alpha_j = -\lambda^T z_f > 0 \tag{18}$$

In case of multiple solutions for λ, one chooses the smallest value of Eq. (18). One observes from the foregoing that the complete solution of the problem is determined by $\Psi(\theta)$ and z_f. Expression (18) suggests the scalar valued variable $\zeta(\theta) = -\lambda^T z(\theta)$. We see from Eqs. (15) and (13) that this variable models fuel consumption

$$\zeta(\theta) = \sum_{j=1}^{i} \alpha_j, \qquad \theta_i \le \theta < \theta_{i+1} \tag{19}$$

and that $\zeta(\theta_f)$ denotes the total cost and satisfies Eq. (18). Specifically, the function ζ additionally satisfies the following: 1) ζ is nonnegative; 2) ζ is monotone nondecreasing; and 3) ζ is a step function.

The preceding work is related to the more conventional primer vector theory mathematically but differs in approach from that theory. To see the relationship, we observe that, in view of Eq. (3), Eqs. (13) and (14) become

$$\alpha_i = 0 \quad \text{or} \quad |q(\theta_i)| = 1 \qquad (i = 1, \dots, n)$$

$$\alpha_i = 0 \quad \text{or} \quad |q(\theta_i)|' = 0$$

for the unspecified values of θ_i in K. These well-known conditions in primer vector theory are fundamentally geometric conditions that must be satisfied by a primer vector for a solution to be optimal. In applying primer vector theory, one adjusts various parameters until all of the required geometric conditions on the primer vector are satisfied, and the graph of the primer vector assumes an ideal shape. This guarantees that a solution is extremal. Our approach is less geometric and more direct. We attempt to solve Eqs. (13) and (14) for λ and the unspecified values of θ_i in K without emphasizing the graph of $|q(\theta)|$. The advantage of our approach is that the problem is reduced to solution of a specified set of nonlinear equations that are quadratic in λ. The disadvantage is that solutions of Eqs. (13) and (14) can be difficult to compute, and efficient computational methods for this system of equations are needed.

B. Bounded-Thrust Problem

We shall consider the state vector $y(\theta)$ to consist of a generalized position vector $x(\theta) \in \Re^m$ and a generalized velocity vector $v(\theta) \in \Re^m$ for each $\theta \in \Theta$ exactly as in the impulsive problem. However, we adopt the convention of Lebesgue measure on Θ and define the class of admissible control functions $\mathcal{U}$ as the set of Lebesgue measurable functions having range in $\Re^m$ such that $|u(\theta)| \le 1$ a.e. on Θ. The abbreviation a.e. on Θ means almost everywhere on Θ, that is, everywhere except on a set of Lebesque measure zero. We are given positive functions β and γ on Θ, and for each $u \in \mathcal{U}$ we define the function $w(\theta) = [0^T, u(\theta)^T]^T$ where 0 is the zero element in $\Re^m$. The bounded-thrust minimization problem is defined as follows:

Find an admissible control function u to minimize the cost function

$$J[u] = \int_{\Theta} \gamma(\theta) |u(\theta)| \, d\theta \tag{20}$$

subject to the differential equation

$$y'(\theta) = A(\theta) y(\theta) + \beta(\theta) w(\theta) \tag{21}$$

which is defined a.e. on Θ and the initial conditions (5) and the terminal conditions (6).

Necessary and sufficient conditions for solution of this problem have been presented under rather general assumptions if $\beta(\theta)$, $\gamma(\theta)$, and the entries in $A(\theta)$ are analytic.[26] For normal nonsingular solutions it is necessary that there exists a vector $\lambda \in \Re^{2m}$ that is either zero or else it defines the primer vector $q(\theta)$ through Eq. (3); this primer vector is nonzero except at most finitely many points of Θ, and

$$u(\theta) = -\frac{q(\theta)}{|q(\theta)|} f(\theta) \tag{22}$$

$$f(\theta) = \begin{cases} 0, & |q(\theta)| < \gamma(\theta)/\beta(\theta) \\ 1, & |q(\theta)| > \gamma(\theta)/\beta(\theta) \end{cases} \tag{23}$$

a.e. on Θ, and

$$y(\theta) = \Phi(\theta) \left[\Phi(\theta_0)^{-1} y_0 + \int_{\theta_0}^{\theta} \beta(\tau) R(\tau) u(\tau) \, d\tau \right] \tag{24}$$

where again $\Phi(\theta)$ represents any fundamental matrix solution associated with Eq. (1).

If we apply the transformation (7) to the trajectory (24), we obtain the relationship

$$z(\theta) = \int_{\theta_0}^{\theta} \beta(\tau) R(\tau) u(\tau) \, d\tau \tag{25}$$

This shows that the control u has a more direct effect on the variable $z(\theta)$ than on $y(\theta)$ and that $z(\theta)$ is constant on a nonthrusting interval.

We can now present a simpler formulation of the bounded-thrust minimization problem involving the variable $z(\theta)$:

Find an admissible control function $u \in \mathcal{U}$ to minimize the cost function (20) subject to

$$z'(\theta) = \beta(\theta)R(\theta)u(\theta) \tag{26}$$

a.e. on Θ and the initial and terminal conditions (9).

If we solve this version of the problem, we will, of course, get the same solution as in the original formulation. To see this, note that, for normal problems, the Hamiltonian associated with Eqs. (25) and (26) is, by abuse of notation,

$$H(\theta) = \gamma(\theta)\ |u(\theta)|\ + l(\theta)^T\beta(\theta)R(\theta)u(\theta)$$

and the adjoint equations, $l'(\theta) = 0$, reveal that $l(\theta)$ is a constant λ. It follows that, for nonsingular solutions, the control function that establishes a minimum of this Hamiltonian pointwise a.e. on Θ is given by Eqs. (22) and (23), providing the same result as in the former version of the problem.

The advantage of viewing the problem in terms of z rather than the original variable y is that coasting intervals do not contribute to the value of z. This pseudostate vector z is determined by thrusting only. For this reason, the effects of the control variable on z should be more apparent than on y, and the boundary conditions should be posed in terms of z instead of y.

To pose the boundary-value problem, we note that a solution of the minimization problem must satisfy Eqs. (3), (22), (23), and (9). Applying these to Eq. (25), we obtain

$$-\int_{\theta_0}^{\theta_f} \frac{\beta(\theta)R(\theta)R(\theta)^T\lambda f(\theta)}{[\lambda^T R(\theta)R(\theta)^T\lambda]^{1/2}}\,\mathrm{d}\theta = z_f \tag{27}$$

This expression presents the mathematical nature of the two-point boundary-value problem very succinctly. The optimal control function is completely determined by $\Psi(\theta)$, β, and z_f. We simply seek a vector $\lambda \in \Re^{2m}$ that satisfies Eq. (27). We iterate on λ through an integral rather than through a differential equation as is usually done in numerical solutions to boundary-value problems. A starting iterative for the numerical solution of Eq. (27) can be obtained from an approximate solution of the related impulsive problem. Using numerical methods for solution of nonlinear equations to solve Eq. (27) for λ, one then gets the primer vector q from Eq. (3) and the control function from Eqs. (22) and (23). The trajectory y can then be obtained by numerical integration of Eq. (21) without inverting the matrix $\Psi(\theta)^T$. If one has an analytical form of the inverse of this matrix, then one can, of course, find the trajectory through Eq. (24), but even then the integral in this expression will almost certainly have to be evaluated numerically. For this reason, there does not appear to be a major advantage in inverting a fundamental matrix solution, even for the problems in which it can easily be done.

We now present some consequences of a necessary condition found recently.[26] The necessary condition, obtained by substituting Eqs. (22) and (3) into Eq. (26) and premultiplying by λ^T, is as follows:

$$\lambda^T z'(\theta) = -\beta(\theta)\,|q(\theta)|\,f(\theta) \le 0 \tag{28}$$

a.e. on Θ and the inequality is strict a.e. on thrusting intervals. Integrating this expression using the initial condition in Eq. (9), we obtain

$$\lambda^T z(\theta) = -\int_{\theta_0}^{\theta} \beta(\tau)\,|q(\tau)|\,f(\tau)\,\mathrm{d}\tau \le 0 \quad (\theta \in \Theta) \tag{29}$$

and the right-hand equality only holds before the first thrusting interval. This shows that the pseudostate trajectory z is completely confined to half of the space $\Re^{2m}$. This can be of value in solving a boundary-value problem. Using Eq. (3) and the terminal condition of Eq. (9), the expression (29) defines

$$\int_{\theta_0}^{\theta_f} \beta(\theta)\,|R(\theta)^T\lambda|\,f(\theta)\,\mathrm{d}\theta + z_f^T\lambda = 0 \tag{30}$$

and, except for the trivial unpowered trajectory where equality holds,

$$\lambda^T z_f < 0 \tag{31}$$

This condition cuts the region of search to solve the boundary-value problem in half. We seek λ from the half-space of $\Re^{2m}$ defined by Eq. (31) that satisfies Eq. (27). We observe that Eqs. (30) and (31) are analogous to Eq. (18).

Proceeding as before, we recall the scalar valued variable $\zeta(\theta) = -\lambda^T z(\theta)$. From Eq. (23) we note that either $\beta(\theta)\,|q(\theta)| > \gamma(\theta)$ or $f(\theta) = 0$ a.e. on Θ; therefore, it follows from Eq. (29) that

$$\zeta(\theta) \ge \int_{\theta_0}^{\theta} \gamma(\tau)f(\tau)\,\mathrm{d}\tau \tag{32}$$

and equality can only hold before the first thrusting interval. As a result we see, in view of Eq. (20), that

$$\zeta(\theta_f) \ge J[u] \tag{33}$$

and equality holds only for the trivial trajectory without thrusting intervals. This restricts the boundary-value problem further. Using Eqs. (31) and (33), the boundary-value problem for trajectories with thrusting intervals becomes the problem of picking $\lambda \in \Re^{2m}$ from the region

$$-z_f^T\lambda > J[u] \tag{34}$$

to satisfy Eq. (27). Although the minimum cost $J[u]$ is not known a priori, crude estimates can be used in practice. Note that, in the impulsive problem, Eq. (34) is replaced by Eq. (18), which establishes one way to obtain a crude estimate.

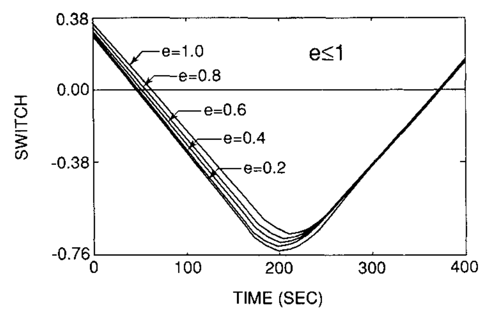

Fig. 1 Switching function for elliptical satellite orbits and parabolic satellite orbit.

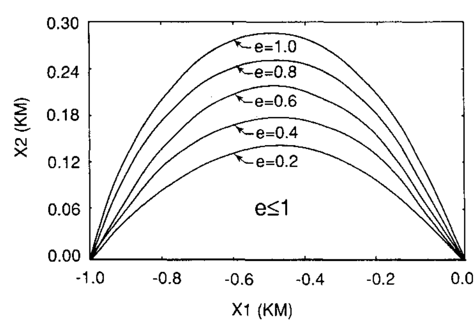

Fig. 2 Spacecraft flight path for elliptical orbits and parabolic satellite orbit.

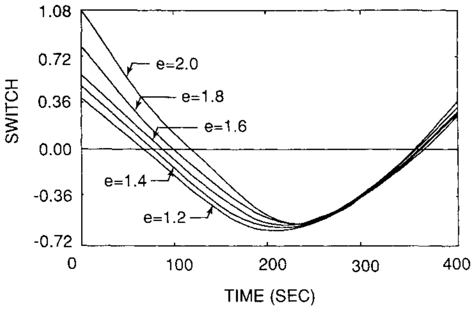

Fig. 3 Switching function for hyperbolic satellite orbits.

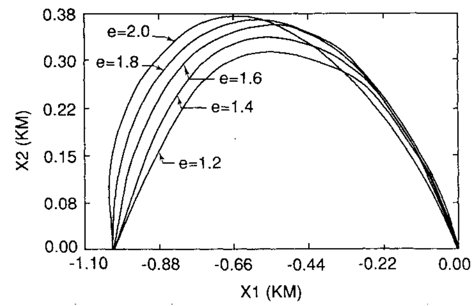

Fig. 4 Spacecraft flight path for hyperbolic satellite orbits.

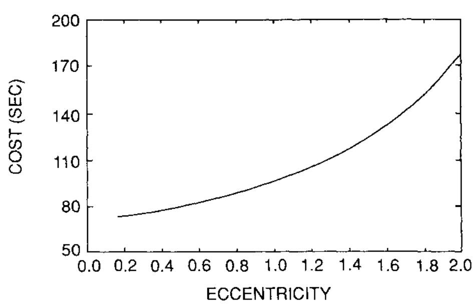

Fig. 5 Optimal fuel expenditure in seconds of burn time at full thrust for increasing satellite orbital eccentricities.

We conclude that for normal nonsingular solutions of the bounded-thrust problem the function ζ satisfies Eq. (33) and 1) ζ is nonnegative; 2) ζ is monotone nondecreasing; and 3) ζ is strictly increasing during thrusting intervals and constant during coasting intervals.

IV. Application to Rendezvous Near Keplerian Orbit

We apply the preceding material to the problem of bounded-thrust spacecraft rendezvous near a satellite in Keplerian orbit.

A. Specific Form for the Keplerian Rendezvous Problem

The transformed equations of motion[12,20] of the spacecraft are as follows:

$$x_1''(\theta) = 2x_2'(\theta) + a_1(\theta)$$

$$x_2''(\theta) = \frac{3}{r(\theta)} x_2(\theta) - 2x_1'(\theta) + a_2(\theta)$$

$$x_3''(\theta) = -x_3(\theta) + a_3(\theta) \tag{35}$$

where θ denotes the true anomaly of the satellite in a Keplerian orbit of eccentricity e, $a(\theta) = [a_1(\theta), a_2(\theta), a_3(\theta)]^T$ is the transformed applied acceleration vector given by

$$a(\theta) = \frac{k}{r(\theta)^3} u(\theta) \tag{36}$$

and

$$r(\theta) = 1 + e \cos\theta \tag{37}$$

The constant k is $L^6 T_m/(\mu^4 m)$ where L is the angular momentum of the satellite divided by its mass, T_m is the maximum spacecraft thrust magnitude, μ is the universal gravitational constant times the mass of the central body of attraction, and m is the mass of the spacecraft. The expenditure of fuel[20] is represented by the cost function

$$J[u] = \int_{\theta_0}^{\theta_f} \frac{|u(\theta)|}{r(\theta)^2} \,\mathrm{d}\theta \tag{38}$$

When Eqs. (35–37) are put in state vector form, we obtain Eq. (21) where

$$A(\theta) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3/r(\theta) & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{39}$$

and $\beta(\theta) = k/r(\theta)^3$. Comparing Eqs. (20) and (38), we see that $\gamma(\theta) = 1/r(\theta)^2$.

We observe that the matrix $A(\theta)$ is not constant. Moreover, the associated fundamental matrix solution is not symplectic. We shall, therefore, not attempt to find and invert a fundamental matrix solution associated with $A(\theta)$.

Avoiding a direct attempt to solve Eq. (21), we turn instead to its adjoint Eq. (4). This leads to the linear equations

$$\begin{aligned} l_1'(\theta) &= 0 \\ l_2'(\theta) &= -[3/r(\theta)]l_4(\theta) \\ l_3'(\theta) &= l_6(\theta) \\ l_4'(\theta) &= -l_1(\theta) + 2l_5(\theta) \\ l_5'(\theta) &= -l_2(\theta) - 2l_4(\theta) \\ l_6'(\theta) &= -l_3(\theta) \end{aligned} \tag{40}$$

The solution of this system of equations is similar to the solution of Eq. (1) with $A(\theta)$ given by Eq. (39). We solve this set of differential equations and find a fundamental matrix solution associated with Eq. (4). The matrix $\Psi(\theta)$ is

$$\begin{bmatrix} 0 & -e & 0 & 0 & 0 & 0 \\ -2 & 3\sin\theta & e^2+3e\cos\theta+2 & 2(e^2+3e\cos\theta+2)I(\theta) - \sin\theta(1+2e\cos\theta)/r(\theta)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\theta & \cos\theta \\ 1 & -[1+r(\theta)]\sin\theta & -r(\theta)^2 & -2r(\theta)^2 I(\theta) & 0 & 0 \\ 0 & -r(\theta)\cos\theta & er(\theta)\sin\theta & 2er(\theta)\sin\theta I(\theta) - \cos\theta/r(\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta \end{bmatrix} \tag{41}$$

where

$$I(\theta) = \int_{\theta_0}^{\theta} \frac{\cos \tau}{r(\tau)^3} \mathrm{d}\tau \qquad (42)$$

The evaluation of this integral for the specific types of orbits has been determined.[22]

In accordance with the ideas presented at the end of Sec. II, we obtain the matrix $R(\theta)^T$ from the last three rows of Eq. (41). In this manner we define the form of the primer vector (3) to be used in the computation of the control function u through Eqs. (22) and (23). The pseudostate vector z can then be obtained through Eq. (25). To solve a specific boundary-value problem, we estimate the minimum value of $J[u]$ and iterate on λ to solve Eq. (27) subject to the restriction of Eq. (34). Starting iteratives satisfy Eq. (34), and if the magnitude of the maximum thrust is very high, they may be obtained from an approximation by an impulsive problem through a solution of Eqs. (13) and (14). Having solved for λ, the pseudostate vector z is of no further use. The vector λ completely determines the optimal control function u from which the entire trajectory can be calculated. The authors have had success with this approach without using new or specialized numerical methods. Newton's method was used to solve for λ, and a variable step size was used for the numerical integration of Eq. (27). These calculations can be performed on a microcomputer.

B. Simulation and Comparison for Various Orbits

The two-point boundary-value problem was solved for a class of satellite orbits having fixed angular momentum and eccentricities varying from $e = 0.2$ through $e = 2$ and a spacecraft with a low thrust-to-mass ratio (0.07853 N/kg). Initially, the satellite follows the spacecraft in the same orbit at a distance of 1 km. The object is fuel-optimal rendezvous with the satellite during a fixed 400-s time interval beginning at perigee. Some comparisons for various Keplerian orbits are presented in Figs. 1–5. Figures 1 and 3 present the switching function $\beta(\theta)\ |q(\theta)|/\gamma(\theta) - 1$ in terms of time in flight. Figures 2 and 4 show the actual flight path $X(\theta) = x(\theta)/r(\theta)$. Figure 5 is a cost comparison for various orbits.

One observes from Figs. 1 and 2 the regular behavior of the switching and flight-path curves for $e \leq 1$ with nearly identical burn times at the end for all cases. Not so for the hyperbolic orbits as indicated by the crossing of the curves in Figs. 3 and 4. From Figs. 1 and 3 one observes an increase in the total burn time as the eccentricity increases. As one might suspect from these curves, the optimal fuel expenditure, as presented in Fig. 5, appears exponential.

V. Conclusions

A transformation of the state vector provides insight into the trajectory optimization problem if the equations of motion are linear. This transformation is based on a fundamental matrix solution of the adjoint system and replaces the need to find a fundamental matrix solution of the original system. For the case of the bounded thrust, the two-point boundary-value problem reduces to the iteration on a vector appearing in a specific integral, and new necessary conditions further restrict the region of search for this vector. This approach avoids the problem of inverting a fundamental matrix solution.

The work is applied to the problem of rendezvous of a bounded-thrust spacecraft with a satellite in Keplerian orbit. A different behavior in the shapes of the flight paths and switching functions is observed for hyperbolic orbits than for orbits of lower eccentricity. Much more fuel is required for the hyperbolic orbits for the cases studied.

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